JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 1, February 2020
http://dx.doi.org/10.14403/jcms.2020.33.1.1

HARMONIC FINSLER METRICS ON SPHERES

Chang-Wan Kim

Abstract. In this paper, it is shown that the reversible harmonic Finsler metrics on spheres must be Riemannian.

1. Introduction

Akbar-Zadeh showed that if a Finsler metric on a compact manifold has constant negative flag curvature, then it is Riemannian, and, if it has zero flag curvature, then it is locally Minkowskian. If a Finsler metric on a compact surfaces has constant positive flag curvature and is, in addition, *reversible*, it is a Riemannian metric by [4]. But, in the non-reversible case, there are many Finsler metrics on 2-spheres with constant positive flag curvature (see, [5]).

A Finsler metric is called the *Zoll* if all of its geodesics are closed and of the same length. The canonical round metric on the compact rank-one symmetric spaces is a Zoll Riemannian metric. However, there exist Zoll Riemannian metrics on spheres which are not round. Contrariwise, a Riemannian metric on the real projective space is a Zoll metric if and only if it has constant sectional curvature since the orientable double cover of a real projective space is a Blaschke sphere. However, this rigidity result fails in the Finsler case (see, [1, 8, 9, 10, 13, 16, 19]).

A Finsler manifold is called *harmonic* if the mean curvature of all geodesic spheres is a function depending only on the radius. It is well-known that the reversible compact harmonic Finsler metrics are Zoll (see, Theorem 3.1). The goal of this paper is to study the reversible harmonic Finsler metrics on spheres.

Theorem 1.1. *The reversible harmonic Finsler metrics on spheres must be Riemannian.*

Received October 30, 2017; Accepted October 17, 2018.
2010 Mathematics Subject Classification: Primary 53C60; Secondary 53B40.
Key words and phrases: Finsler metrics, Blaschke, harmonic, spheres.

The corresponding question about reversible harmonic Finsler metrics on compact rank-one symmetric spaces remains open at this writing, since an essential component of the proof for harmonic Finsler spheres due to Shen (see, Theorem 2.1) has not yet been generalized to compact rank-one symmetric spaces (cf. [11]).

2. Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [18], for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties

- F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;
- $F(ty) = tF(y)$, for all $t > 0$, $y \in T_xM$;
- for each $y \in T_xM \setminus \{0\}$, the following quadratic g_y is an inner product in T_xM,

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s=t=0}.$$

A manifold M endowed with a Finsler structure will be called a Finsler manifold. Note that we never require smoothness at the zero section. Finsler metrics for which all norms $F(x, \cdot)$ are symmetric will be called *reversible* Finsler metrics.

The Chern connection on a Finsler manifold M is defined by the unique set of local 1-forms $\{\omega_j^{\ i}\}_{1\le i,j\le n}$ on $\widetilde{TM}$ such that

$$\begin{aligned} d\omega^i &= \omega^j \wedge \omega_j^{\ i}, \\ dg_{ij} &= g_{kj}\omega_i^{\ k} + g_{ik}\omega_j^{\ k} + 2A_{ijk}\omega_n^{\ k}, \text{ where } A_{ijk} = \frac{\partial g_{ij}}{\partial y^k}. \end{aligned}$$

Define the set of local curvature forms $\Omega_j^{\ i}$ by

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i}.$$

Then one can write

$$\Omega_j^{\ i} = \frac{1}{2} R_j^{\ i}{}_{kl}\, \omega^k \wedge \omega^l + P_j^{\ i}{}_{kl}\, \omega^k \wedge \omega^{n+l}.$$

Define the curvature tensor R by $R(U, V)W = u^k v^l w^j R_j^{\ i}{}_{kl} E_i$, where $U = u^i E_i, V = v^i E_i, W = w^i E_i$ are vectors in the pull-back bundle π^*TM of TM by $\pi : \widetilde{TM} \to M$. For a fixed $v \in T_xM$ let $\gamma_v(t)$ be the geodesic from $\gamma_v(0) = x$ with $\dot{\gamma}_v(0) = v$. Along $\gamma_v(t)$, we have the osculating Riemannian metrics

$$g_{\dot{\gamma}_v(t)} := g\big(\gamma_v(t), \dot{\gamma}_v(t)\big)$$

in $T_{\gamma_v(t)}M$. Define the flag curvature

$$R_{\dot{\gamma}_v(t)} : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$$

by

$$R_{\dot{\gamma}_v(t)}\big(u(t)\big) := R\big(U(t), V(t)\big)V(t),$$

where $U(t) = (\hat{\gamma}_v(t); u(t)), V(t) = (\hat{\gamma}_v(t); \gamma_v(t)) \in \pi^*TM$. The flag curvature is independent of connections, that is, the term appears in the second variation of arc length, thus is of particular interest to us. We remark that if F is Riemannian, then the flag curvature coincides with the sectional curvature. Then the Ricci curvature is defined by

$$\mathrm{Ric}(v) := \sum_{i=1}^{n} g_v\big(R_v(e_i), e_i\big), v \in T_xM,$$

where $\{e_i\}_{i=1}^n$ is a g_v-orthonormal basis for T_xM.

Shen ([17]) defined the S-curvature $S(v)$ what measures the average change of $(T_xM, F(x, \cdot))$ in the direction $v \in T_xM$. We say $|S| \leq \delta$ if $|S(v)| \leq \delta F(v)$ for all $v \in \widetilde{TM}$. An important property is that $S = 0$ for Finsler manifolds modeled on a single Minkowski space. In particular, $S = 0$ for Berwald spaces. Locally Minkowski spaces and Riemannian spaces are all Berwald spaces.

By [6, Sect. 5.5], there is only one reasonable notion of the volume for Riemannian manifolds. However, the situation is different in Finsler geometry. The Finsler volume can be defined in various ways and essentially different results may be obtained, e.g., [6, 17]. Therefore, it is an interesting and important problem to investigate the relations between the volumes and the geometric properties on a Finsler manifold.

The Busemann-Hausdorff volume vol^{bh} of a Finsler space is that multiple of the Lebesgue measure for which the volume of the unit ball equals the volume of Euclidean unit ball. Using Brunn-Minkowski theory, Busemann proved that the Busemann-Hausdorff volume of an n-dimensional Finsler space equals its n-dimensional Hausdorff measure. Hence, from the viewpoint of metric geometry, this is a very natural definition.

For a constant $\lambda \in \mathbb{R}$ and $\delta \geq 0$, put

$$V_{\lambda,\delta}(r) := \alpha(n-1) \cdot \int_0^r e^{\delta t} s_\lambda(t)^{n-1} dt,$$

where $s_\lambda(t)$ denotes the unique solution to $z'' + \lambda z = 0$ with $z(0) = 0, z'(0) = 1$, and $\alpha(n-1)$ the volume of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

A Finsler volume used is the Busemann-Hausdorff volume vol^{bh}, with respect to which Shen firstly obtained the following Bishop-Gromov type volume comparison theorem in [17].

THEOREM 2.1. *Let (M, F) be a complete n-dimensional Finsler manifold. If $\mathrm{Ric} \geq (n-1)\lambda$, $|S| \leq \delta$, then for all $x \in M$ and for any $0 < r < R$ we have*

$$\frac{\mathrm{vol}^{bh}(B(x,r))}{V_{\lambda,\delta}(r)} \geq \frac{\mathrm{vol}^{bh}(B(x,R))}{V_{\lambda,\delta}(R)}.$$

Furthermore, we have the equality if and only if any Jacobi field $J_u(t)$ along γ_v has the form, $J_u(t) = s_\lambda(t)u(t)$, where $u = u(t)$ is a parallel vector field along γ_v.

Another volume that is used frequently in Finsler geometry is the so-called Holmes-Thompson volume. The Holmes-Thompson volume vol^{ht} of n-dimensional compact Finsler manifold (M, F) is the symplectic volume of the unit co-disc bundle divided by the volume of the Euclidean unit ball of dimension n. In the case of Riemannian metrics, all unit tangent spaces are isometric to the Euclidean spheres, and we have $\mathrm{vol}^{ht}(M) = \mathrm{vol}^{bh}(M)$. On the other hand, in a general Finsler metric, unit tangent spaces may not be isometric to each other, and hence one can not expect the equality. We instead have the following theorem.

THEOREM 2.2. *([7]) Let (M, F) be an n-dimensional compact reversible Finsler manifold. Then we have*

$$\mathrm{vol}^{ht}(M) \leq \mathrm{vol}^{bh}(M).$$

with equality if and only if (M, F) is a Riemannian metric

There exist counterexamples to the inequality when F is nonreversible, e.g., [15].

A Zoll Finsler manifold is called a $C_{2\pi}$-manifold, if all geodesics are closed and of the same length 2π. The following statements are standard whose proofs can be found also in [2, 14].

THEOREM 2.3. *If (M, F) be an n-dimensional Finsler $C_{2\pi}$-manifold, then the ratio*

$$i(M) = \frac{\mathrm{vol}^{ht}(M, F)}{\mathrm{vol}^{ht}(\mathbb{S}^n, g_0)}$$

is an integer.

Here, and in what follows, $(\mathbb{S}^n, g_0)$ is the canonical Riemannian sphere $\mathbb{S}^n$ of radius 1 in $\mathbb{R}^{n+1}$.

REMARK 2.4. Under the assumption of Theorem 2.3, if M is homeomorphic to one of the compact rank-one symmetric spaces $\mathbb{P}$, i.e., $\mathbb{S}^n$, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, $\mathbb{C}aP^2$, Weinstein, Yang, and Reznikov showed that $\mathrm{vol}^{ht}(M, F) = \mathrm{vol}^{ht}(\mathbb{P}, g_0)$.

3. Harmonic Finsler manifolds

A reversible Finsler manifold is called a *Blaschke* manifold, if every minimal geodesic of length less than the diameter is the unique shortest path between any of its points. Equivalently, for which all cut loci are round spheres of constant radius and dimension. For a reversible Blaschke Finsler manifold the exponential map restricted to the unit tangent sphere defines a great sphere foliation. Since every great sphere foliation of sphere is homeomorphic to a Hopf fibration, simply connected reversible Blaschke Finsler manifolds are actually homeomorphic to compact rank-one symmetric spaces (cf. [12]).

The mean curvature $m_t(v)$ of geodesic sphere of radius t about geodesic $\gamma_v(t)$ has following Taylor expansion

$$m_t(v) = \frac{n-1}{t} - S(v) - \frac{1}{3}\Big(\mathrm{Ric}(v) + 3\dot{S}(v)\Big)t + O(t),$$

where S is S-curvature. A Finsler manifold is called *harmonic* if the mean curvature $m_t(v)$ of all geodesic spheres is a function depending only on the radius t. Hence the harmonic Finsler manifolds have Einstein metrics and zero S-curvature.

A historical break in the theory of harmonic Riemannian manifolds was made by Allamigeon when he proved the following: A simply connected harmonic Riemannian manifold is either diffeomorphic to Euclidean space or is a Blaschke Finsler manifold. The following theorem is to put them in a Finsler-geometric setting. For the sake of completeness we sketch the proof.

THEOREM 3.1. *A simply connected reversible harmonic Finsler manifold M is either diffeomorphic to Euclidean space or is a Blaschke Finsler manifold.*

Proof. Suppose there is no conjugate points. Then exponential map is a covering map and since M is simply connected, a diffeomorphism. So take a $0 \neq v_0 \in T_xM$ and an $r_0 \in \mathbb{R}$ such that the first conjugate point along γ_{v_0} is $\gamma_{v_0}(r_0)$. Then the first conjugate point along γ_v is $\gamma_v(r_0)$ for all $v \in T_xM$, since the mean curvature is radial. Note that r_0 is the same for every point in M. This means that M is a Blaschke manifold by the Allamigeon-Warner theorem, cf. [3, Corollary 5.31]. $\square$

Now we are ready to prove main theorem using Theorems 2.1, 2.2, 2.3 and Remark 2.4.

THEOREM 3.2. *A reversible harmonic Finsler metric F on sphere M is Riemannian.*

Proof. Up to a scaling of the metric, we assume that the reversible harmonic Finsler sphere (M, F) is an n-dimensional Blaschke $C_{2\pi}$-manifold. By Theorem 2.3 and Remark 2.4, we have $\mathrm{vol}^{ht}(M, F) = \mathrm{vol}^{ht}(\mathbb{S}^n, g_0)$. On the other hand, (M, F) has constant Ricci curvature $(n-1)$ and zero S-curvature, and by Theorem 2.1, we obtain $\mathrm{vol}^{bh}(M, F) \leq \mathrm{vol}^{bh}(\mathbb{S}^n, g_0)$. Thus we conclude

$$\begin{aligned} \mathrm{vol}^{bh}(\mathbb{S}^n, g_0) &= \mathrm{vol}^{ht}(\mathbb{S}^n, g_0) \\ &= \mathrm{vol}^{ht}(M, F) \\ &\leq \mathrm{vol}^{bh}(M, F) \\ &\leq \mathrm{vol}^{bh}(\mathbb{S}^n, g_0). \end{aligned}$$

We note that the third line is obtained from Theorem 2.2, and hence we obtain $\mathrm{vol}^{ht}(M, F) = \mathrm{vol}^{bh}(M, F)$. Then by the equality case of Theorem 2.2, F is a Riemannian metric. $\square$

References

[1] A. Abbondandolo, B. Bramham, U. L. Hryniewicz, and P. A. S. Salomão, *Sharp systolic inequalities for Riemannian and Finsler spheres of revolution, preprint*, available at arXiv:math/1808.06995 (2018).

[2] J. C. Álvarez Paiva, *Asymmetry in Hilbert's fourth problem*, preprint, available at arXiv:math/1301.2524 (2013).

[3] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete **93**, Springer-Verlag 1978.

[4] R. L. Bryant, *Geodesically reversible Finsler 2-spheres of constant curvature*, Inspired by S. S. Chern, Nankai Tracts Math., **11** (2006), 95–111.

[5] R. L. Bryant, P. Foulon, S. Ivanov, V. S. Matveev, and W. Ziller, *Geodesic behavior for Finsler metrics of constant positive flag curvature on S^2*, to appear J. Differential Geom.

[6] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics **33**, American Mathematical Society 2001.

[7] C. E. Durán, *A volume comparison theorem for Finsler manifolds*, Proc. Amer. Math. Soc., **126** (1998), 3079–3082.

[8] N. Innami, Y. Itokawa, T. Nagano, and K. Shiohama, *Blaschke Finsler manifolds and actions or projective Randers changes on cut loci*, Trans. Amer. Math. Soc., **371** (2019), 7433–7450.

[9] S. Ivanov, *Filling minimality of Finslerian 2-discs*, Proc. Steklov Inst. Math., **273** (2011), 176–190.

[10] K. Kiyohara, S. V. Sabau, and K. Shibuya, *The geometry of a positively curved Zoll surface of revolution*, Int. J. Geom. Methods Mod. Phys., **16** (2019), no. 02, 1941003.

[11] C.-W. Kim, *Locally symmetric positively curved Finsler spaces*, Arch. Math., **88** (2007), 378–384.

[12] ———, *Harmonic Finsler manifolds with minimal horospheres*, Commun. Korean Math. Soc., **33** (2018), 929–933.

[13] ———, *Blaschke Finsler metrics on spheres*, Int. J. Geom. Methods Mod. Phys., **16** (2019), no. 9, 9 pages.

[14] C.-W. Kim and J.-W. Yim, *Finsler manifolds with positive constant flag curvature*, Geom. Dedicata, **98** (2003), 47–56.

[15] M. Matsumoto, *Theory of curves in tangent planes of two-dimensional Finsler spaces*, Tensor (N.S.) **37** (1982), 35–42.

[16] S. Sabourau, *Strong deformation retraction of the space of Zoll Finsler projective planes*, J. Symplectic Geom., **17** (2019), 443–476.

[17] Z. Shen, *Volume comparison and its applications in Riemannian-Finsler geometry*, Adv. Math., **128** (1997), 306–328.

[18] Y.-B. Shen and Z. Shen, *Introduction to modern Finsler geometry*, World Scientific 2006.

[19] K. Shiohama and B. Tiwari, *The global study of Riemannian-Finsler geometry*, Geometry in History, Springer-Verlag, (2019), 581–621.

Division of Liberal Arts and Sciences
Mokpo National Maritime University
Mokpo 58628, Korea
E-mail: cwkim@mmu.ac.kr